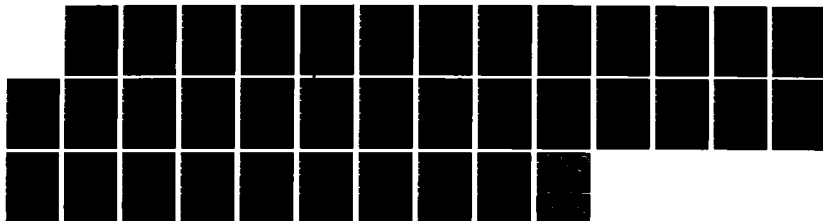
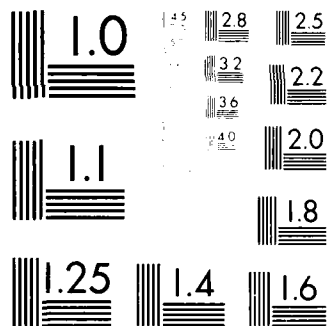


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**A General
Software Availability/Reliability Model:
Numerical Exploration
via the Matrix Laguerre Transform**

by

Yasushi Masuda†
J.G. Shanthikumar‡
Ushio Sumita†

Working Paper Series No. QM8425

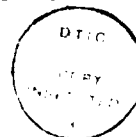
December 1984

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ABSTRACT

In this paper, a new software availability/reliability model is developed where life-times and repair times have general system-state-dependent distributions. Multiple errors may be introduced or removed through repairs. The model is formulated as a multivariate Markov process and contains many other models appeared in the literature as special cases. The exponentiality assumption prevalent in the literature is totally eliminated. Expressions of various performance measures of practical interest combining availability and reliability of the software system at time t are derived. Using the matrix Laguerre transform of Sumita(1984), corresponding computational procedures are also developed. A numerical example is given, demonstrating speed, accuracy and stability of these procedures.

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§.0 Introduction

The price performance revolution of computer hardware has been dramatic, while the cost of labor has been steadily increasing. Consequently the production and maintenance cost of software, in contrast to that of hardware, has been rapidly growing and has become one of central issues in system design. Dating back to the late 60's substantial research efforts have been devoted to the study of software failure phenomenon and the prediction of software performance. Two recent survey papers by Ramamoothy and Bastani(1982) and Shanthikumar(1983) contain approximately 150 refereneces on the issues.

Until recently, however, the effect of multiple error generation and removal from the system during the repair has not been properly incorporated in the literature. Kremer(1983) has derived a software reliability model where the number of software system increases or decreases by at most one during repairs. He has provided performance measures for this model using the results available for non-homogeneous birth-death process. Sumita and Shanthikumar(1984) have developed a general Markov chain model where multiple errors may be introduced or removed from the system during repairs. Assuming that the software failure rate is propotional to the number of software errors present in the system, expressions for various software reliability measures of interest are derived and corresponding computational procedures are developed.

The exponentiality assumption employed in the model of Sumita and Shanthikumar(1984) is rather restrictive. To preserve the Markov chain property, for example, the software repair time is assumed to be negligible, ignoring the availability of software system at time t . The problem of a combined availability/reliability analysis of software system was addressed by Shanthikumar(1984) in a limited model where the number of

software errors is either unchanged or reduced by one during the repairs. The purpose of this paper is to develop a general multivariate Markov model for a software system with multiple software error generation and removal during the repairs. The exponentiality assumption is totally eliminated and general repair time distributions are explicitly incorporated. Expressions of various performance measures combining availability and reliability of software system at time t are derived. Using the matrix Laguerre transform of Sumita(1984) as a key tool, corresponding computational procedures are also developed. Many other models appeared previously in the literature can be treated as special cases of this model.

In Section 1, we develop a new software availability/reliability model having system-state-dependent lifetimes and repair times. Multiple errors may be introduced or removed through repairs. The model is formulated as a multivariate Markov process and does not require exponentiality at all. By studying the probabilistic flow in the corresponding state space, various time dependent entities are analyzed. In Section 2, expressions of many performance measures are derived in terms of these probabilistic entities. Performance measures combine time dependent availability and reliability of the software system. Computational procedures for evaluating these performance measures are developed in Section 3, using the matrix Laguerre transform of Sumita(1984). Section 4 is devoted to numerical implementation of the procedures demonstrating their speed, accuracy and stability.

§.1 Model Description and Analysis

We consider a software system which contains several software errors. These software errors cause software system failure time to time. Upon failure, the software system is repaired. During repairs, multiple software errors may be introduced or removed. More formally, the following assumptions are incorporated in our model.

(AS1) The maximum number of software errors in the software system is limited to K , $0 < K < \infty$. (If the maximum number of errors exceeds K , then the performance of software system becomes intolerable and the system would be discarded.)

(AS2) At time $t = 0$, the software system starts functioning and there are N errors in the software. Here N is a discrete nonnegative random variable with probability vector $\underline{b}^T = (b_0, \dots, b_K)$ where $b_i = P[N = i]$, $i = 0, 1, \dots, K$.

(AS3) If there are n software errors upon completion of a repair, then the time until next software system failure has c.d.f. $A_n(x)$, $n = 0, 1, 2, \dots, K$. In particular $A_0(x) = 0$, $x \geq 0$, i.e. if there is no error in the software system, then there will be no software system failures. It is assumed that, for $n \geq 1$, $A_n(x)$ is absolutely continuous with p.d.f. $a_n(x)$ and hazard function $\eta_n(x) = a_n(x) / \bar{A}_n(x)$ where $\bar{A}_n(x) = 1 - A_n(x)$.

(AS4) If there are n errors at the begining of a repair, then the repair time has a c.d.f. $R_n(x)$, $n = 1, 2, \dots, K$. It is assumed that $R_n(x)$ is absolutely continuous with p.d.f. $r_n(x)$ and hazard function $\zeta_n(x) = r_n(x) / \bar{R}_n(x)$.

(AS5) The probability that there are n errors remaining in the software system immediately after a repair given that there were k errors in the software system just before the begining of the repair is p_{kn} . For notational convenience, we define $p_{00} = 1$, $p_{0n} = 0$, $n \geq 1$.

(AS6) All software system lifetimes and repair times are mutually independent.

This model can be expressed as a multivariate Markov process in terms of the following stochastic processes:

$$(1.1) \quad I(t) = \begin{cases} 0, & \text{if the software system is under repair at time } t. \\ 1, & \text{if the software system is functioning at time } t. \end{cases}$$

$$(1.2) \quad M(t) = \text{the number of software system failures occurred in } [0, t).$$

$$(1.3) \quad N(t) = \text{the number of errors in software system at time } t.$$

$$(1.4) \quad X(t) = \text{the elapsed time since the last transition of } I(t), \text{ i.e.,}$$

$$X(t) = t - \tau, \text{ where } \tau = \sup_{0 \leq x \leq t} \{x : |I(x+) - I(x-)| = 1\}$$

From (AS2), one has $X(0+) = 0$. For notational convenience we assume that $I(0-) = 0$ and $I(0+) = 1$. Clearly the multivariate process $\{I(t), M(t), N(t), X(t)\}$ is Markov. The state space of the multivariate process and its typical transition behavior are depicted in Figure 1.1.

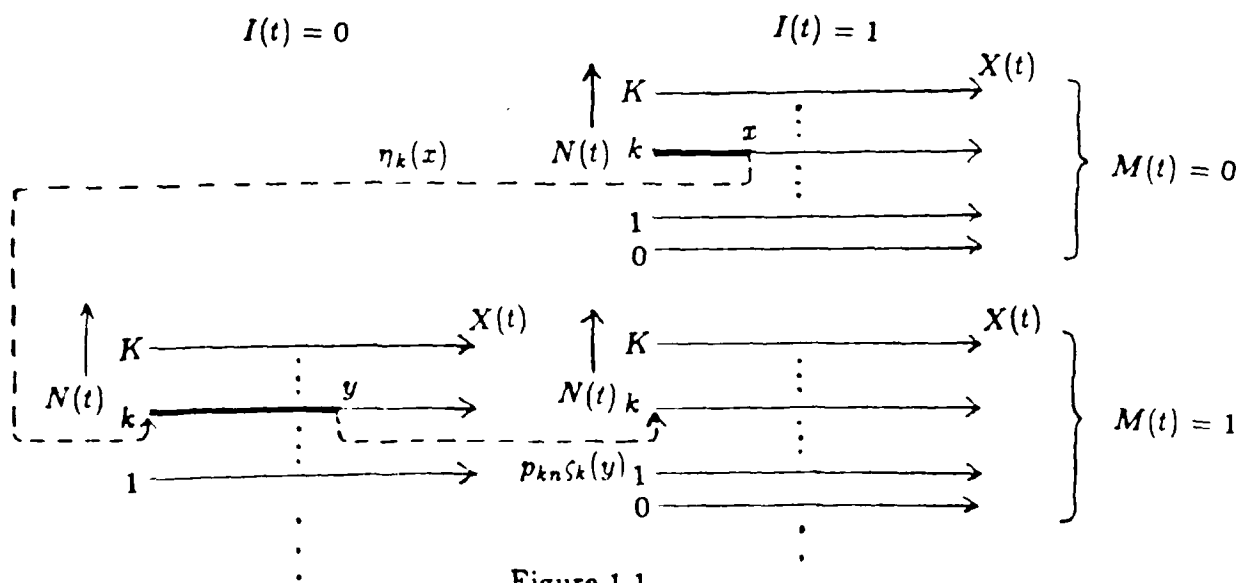


Figure 1.1

Let

$$(1.5) \quad F_{i,m,n}(x,t) = P\{I(t) = i, M(t) = m, N(t) = n, X(t) \leq x\},$$

and define

$$(1.6) \quad f_{i,m,n}(x,t) = \frac{\partial}{\partial x} F_{i,m,n}(x,t).$$

The partial differentiability $f_{i,m,n}(x,t) = \frac{\partial}{\partial x} F_{i,m,n}(x,t)$ can be shown through a renewal argument (see, e.g., Çinlar(1969)), except the case $m = 0$ for which generalized densities $f_{1,0,n}(x,t) = b_n \delta(t-x) \bar{A}_n(x)$ are involved. As we will see in Section 2, all performance measures of practical interest for the study of availability/ reliability of the software system can be expressed in terms of (1.5) and (1.6). In the remainder of this section, we derive transform results of (1.6) by applying the state space method of Keilson and Kooharian(1960, 1962). The corresponding computational procedures will be discussed in Section 3.

We observe from Figure 1.1 that for the process to be at (i, m, n, x) at time t where $0 < x < t$, the process must have entered $(i, m, n, 0)$ at time $t - x$ and has remained in the state (i, m, n, \cdot) for the length of x . We note from (AS3) that $A_0(x) = 0, x \geq 0$. Therefore once the process enters $(1, m, 0, \cdot)$, it remains there. Hence one has

$$(1.7) \quad f_{0,m,n}(x,t) = f_{0,m,n}(0+, t-x) \bar{R}_n(x), \quad m, n \geq 1$$

and

$$(1.8) \quad f_{1,m,n}(x,t) = f_{1,m,n}(0+, t-x) \bar{A}_n(x), \quad m, n \geq 0.$$

By a similar argument, boundary conditions can be found as

$$(1.9) \quad f_{i,m,n}(x,0) = 0, \quad i = 0, 1, \quad m, n \geq 0, \quad x > 0,$$

and

$$(1.10) \quad f_{0,m,n}(0+,t) = \int_0^t f_{1,m-1,n}(x,t) \eta_n(x) dx, \quad m, n \geq 1.$$

Futhermore one has

$$(1.11) \quad \begin{cases} f_{1,m,n}(0+,t) = \sum_k p_{kn} \int_0^t f_{0,m,k}(x,t) \zeta_k(x) dx, & m \geq 1, n \geq 0, \\ f_{1,0,n}(0+,t) = b_n \delta(t), & n \geq 0. \end{cases}$$

Substituting (1.7) and (1.8) into (1.10) and (1.11), one obtains for $t > 0$,

$$(1.12) \quad f_{0,m,n}(0-,t) = \int_0^t f_{1,m-1,n}(0+,t-x) a_n(x) dx, \quad m \geq 1, \quad n \geq 1,$$

and

$$(1.13) \quad \begin{cases} f_{1,m,n}(0+,t) = \sum_k p_{kn} \int_0^t f_{0,m,k}(0+,t-x) r_k(x) dx, & m \geq 1, n \geq 0 \\ f_{1,0,n}(0+,t) = b_n \delta(t), & n \geq 0. \end{cases}$$

Let

$$(1.14) \quad \begin{cases} \hat{\varphi}_{i,m,n}(0+,s) = \int_0^\infty e^{-st} f_{i,m,n}(0+,t) dt \\ \hat{\varphi}_{i,m,n}(w,s) = \int_0^\infty \int_0^\infty e^{-wx-st} f_{i,m,n}(x,t) dx dt. \end{cases}$$

Equation (1.12) and (1.13) can be best described in terms of the transform of (1.14) using the matrix notation. For notational convenience, we introduce

$$(1.15) \quad f_{0,m,0}(x,t) \stackrel{\text{def}}{=} 0, \quad m \geq 0, \quad x, t \geq 0$$

and define the transform vectors

$$(1.16) \quad \begin{cases} \hat{\underline{\varphi}}_{i,m}^T(0+,s) = [\hat{\varphi}_{i,m,0}(0+,s), \dots, \hat{\varphi}_{i,m,K}(0+,s)] \\ \hat{\underline{\varphi}}_{i,m}^T(w,s) = [\hat{\varphi}_{i,m,0}(w,s), \dots, \hat{\varphi}_{i,m,K}(w,s)]. \end{cases}$$

Let $\text{diag}\{c_0, \dots, c_K\}$ be a $(K+1) \times (K+1)$ diagonal matrix whose n -th diagonal element is c_{n-1} . We define $\underline{\alpha}_D(s) = \text{diag}\{0, \alpha_1(s), \dots, \alpha_K(s)\}$ and $\underline{\rho}_D(s) = \text{diag}\{0, \rho_1(s), \dots, \rho_K(s)\}$

where $\alpha_n(s) = \int_0^\infty e^{-sz} a_n(x) dx$ and $\rho_n(s) = \int_0^\infty e^{-sz} r_n(x) dx$. One then sees from (1.12) and (1.13) that

$$(1.17) \quad \hat{\varphi}_{0,m}^T(0+, s) = \hat{\varphi}_{1,m-1}^T(0+, s) \underline{\alpha}_D(s), \quad m \geq 1,$$

and

$$(1.18) \quad \begin{cases} \hat{\varphi}_{1,m}^T(0+, s) = \hat{\varphi}_{0,m}^T(0+, s) \underline{\rho}_D(s) \underline{P}, & m \geq 1, \\ \hat{\varphi}_{1,0}^T(0+, s) = \underline{b}^T, \end{cases}$$

where $\underline{P} = (p_{ij})$. Hence one concludes that

$$(1.19) \quad \hat{\varphi}_{0,m}^T(0+, s) = \underline{b}^T \left(\underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P} \right)^{m-1} \underline{\alpha}_D(s), \quad m \geq 1,$$

and

$$(1.20) \quad \hat{\varphi}_{1,m}^T(0+, s) = \underline{b}^T \left(\underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P} \right)^m, \quad m \geq 0.$$

Finally from (1.7), (1.8), (1.19) and (1.20), one has

$$(1.21) \quad \hat{\varphi}_{0,m}^T(w, s) = \frac{1}{s+w} \hat{\varphi}_{0,m}^T(0+, s) (\underline{I} - \underline{\rho}_D(w+s)), \quad m \geq 1$$

and

$$(1.22) \quad \hat{\varphi}_{1,m}^T(w, s) = \frac{1}{s+w} \hat{\varphi}_{1,m}^T(0+, s) (\underline{I} - \underline{\alpha}_D(w+s)), \quad m \geq 0.$$

We note that the spectral radius of $(\underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})$ is strictly less than one for $\operatorname{Re}(s) > 0$ and $\sum_{m=0}^\infty (\underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})^m = (\underline{I} - \underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})^{-1}$. Hence

$$(1.23) \quad \sum_{m=1}^\infty \hat{\varphi}_{0,m}^T(0+, s) = \underline{b}^T (\underline{I} - \underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})^{-1} \underline{\alpha}_D(s),$$

and

$$(1.24) \quad \sum_{m=0}^\infty \hat{\varphi}_{1,m}^T(0+, s) = \underline{b}^T (\underline{I} - \underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})^{-1}.$$

Let $\underline{1}$ be a vector of length $(K + 1)$ having all elements equal to one. It can be readily seen from (1.21) through (1.24) that

$$\begin{aligned}
 & \sum_{m=0}^{\infty} \left\{ \hat{\underline{\varphi}}_{0,m+1}^T(w, s) \underline{1} + \hat{\underline{\varphi}}_{1,m}^T(w, s) \underline{1} \right\} |_{w=0} \\
 &= \frac{1}{s} \underline{b}^T (\underline{I} - \underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})^{-1} (\underline{I} - \underline{\alpha}_D(s) \underline{\rho}_D(s)) \underline{1} \\
 &= \frac{1}{s} \underline{b}^T (\underline{I} - \underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})^{-1} (\underline{I} - \underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P}) \underline{1} \\
 &= \frac{1}{s} \underline{b}^T \underline{1} = \frac{1}{s},
 \end{aligned}$$

as expected.

This model contains the model of Sumita and Shanthikumar (1984) as a special case where $A_n(x) = 1 - e^{-n\lambda x}$ and $R_n(x) = 1$ for $x > 0$. Hence many other models contained in Sumita and Shanthikumar(1984) are also contained in this model. Shanthikumar(1984) has discussed another special case of this model where $p_{ij} > 0$ if and only if $j = i$ or $j = i - 1$.

§.2 Performance of the Software System: Availability/Reliability Measures

In this section we introduce various availability/reliability measures for the study of performance of the software system. Based on the analytical results of Section 1, we derive explicitly expressions of these availability/reliability measures.

A natural availability/reliability measure is the joint probability of $I(t)$ (whether the software system is working or not at time t), $M(t)$ (the number of software system failures occurred in $[0, t)$), and $N(t)$ (the number of errors in the software system at time t). One has

$$(2.1) \quad P[I(t) = i, M(t) = m, N(t) = n] = F_{i,m,n}(+\infty, t).$$

The joint probability of $I(t)$ and $N(t)$ is also of interest. For notational convenience, we define $F_{0,0,n}(+\infty, t) = 0$, $n \geq 0$, $t \geq 0$. One then sees that

$$(2.2) \quad P[I(t) = i, N(t) = n] = \sum_{m=0}^{\infty} F_{i,m,n}(+\infty, t).$$

The availability(unavailability) measure is the probability that the software system is functioning(not functioning) at time t , given by

$$(2.3) \quad P[I(t) = i] = \sum_{n=0}^K \sum_{m=0}^{\infty} F_{i,m,n}(+\infty, t).$$

The time until complete debugging is an important performance measure. Let $X_{N(0),0}$ be the time required for completely eliminating all errors in the software system given that $N(0+) = k$ with probability b_k . We denote the distribution function of $X_{N(0),0}$ by $S_{N(0),0}(x) = P[X_{N(0),0} \leq x]$. Since the states $(1, m, 0, \cdot)$ are absorbing, one finds that

$$(2.4) \quad S_{N(0),0}(t) = P[N(t) = 0] = \sum_{m=0}^{\infty} F_{1,m,0}(+\infty, t).$$

In the study of performance of software systems it is of more practical interest to predict the software availability/reliability based on the past observation of the system. In order to obtain performance measures in this context, we impose the following history of the system:

(CD) A repair has just been completed at time t_0 and there have been m software failures in $[0, t_0]$.

The probability distribution of the number of errors remaining in the software system at time t_0 given (CD) plays a crucial role in this analysis. Let $\beta_{k|t_0, m} = P\{N(t_0) = k | (CD)\}$ and define $\underline{\beta}_{|t_0, m}^T = [\beta_{0|t_0, m}, \beta_{1|t_0, m}, \dots, \beta_{K|t_0, m}]$. One then finds that

$$(2.5) \quad \underline{\beta}_{|t_0, m}^T = \underline{f}_{1, m}^T(0, t_0) / \underline{f}_{1, m}^T(0, t_0) \underline{1}.$$

The vector $\underline{\beta}_{|t_0, m}^T$ fully describes the state of the system at time t_0 under (CD), which then provides an initial state probability vector for the system behavior after time t_0 . To emphasize the dependence of $F_{i, m, n}(x, t)$ on the initial distribution \underline{b}^T we write

$$(2.6) \quad F_{i, m, n}(x, t | \underline{b}) =$$

$$P\{I(t) = i, M(t) = m, N(t) = n, X(t) \leq x | N(0+) = k \text{ with probability } b_k\}.$$

Given (CD), the joint probability of the system availability/unavailability and the number of errors in the software system at time $t_0 + \tau$ is then obtained as

$$(2.7) \quad P\{I(t_0 + \tau) = i, N(t_0 + \tau) = n | (CD)\} = \sum_{j=0}^{\infty} F_{i, j, n}(+\infty, \tau | \underline{\beta}_{|t_0, m}).$$

Similarly the joint probability of the system availability/unavailability and the number of software failures that may occur in $[t_0, t_0 + \tau)$ under (CD) is given by

$$(2.8) \quad P\{I(t_0 + \tau) = i, M(t_0 + \tau) - m = j | (CD)\} = \sum_{n=0}^K F_{i, j, n}(+\infty, \tau | \underline{\beta}_{|t_0, m}).$$

Let $Y_{N(t_0),0}$ denote the random duration required for completely eliminating the software errors given (CD). The corresponding distribution function is defined by $G_{N(t_0),0}(\tau) = P\{Y_{N(t_0),0} \leq \tau\}$. It can be readily seen that

$$(2.9) \quad G_{N(t_0),0}(\tau) = \sum_{j=0}^{\infty} F_{1,j,0}(+\infty, \tau | \underline{\beta}_{|t_0,m}).$$

Finally an important reliability measure is the time until the next software system failure given (CD). We denote this random variable by $T_{|t_0,m}$. The survival function $\bar{W}_{|t_0,m}(\tau) = P\{T_{|t_0,m} > \tau\}$ is then given by

$$(2.10) \quad \bar{W}_{|t_0,m}(\tau) = \sum_{k=0}^K \beta_{k|t_0,m} \bar{A}_k(\tau).$$

We note that $T_{|t_0,m}$ is dishonest and $\bar{W}_{|t_0,m}(\tau) \rightarrow \beta_{0|t_0,m}$ as $\tau \rightarrow \infty$.

It should be noted that the marginal process $N(t)$ is absorbing and all the performance measures described above are time dependent. One time independent performance measure of interest is the distribution of the number of software system failures that occur before all software errors are completely eliminated. We denote the corresponding random variable by D . One then sees that $d_m = P\{D = m\} = F_{1,m,0}(+\infty, +\infty) = \lim_{s \rightarrow 0+} s \hat{\phi}_{1,m,0}(0, s)$. Hence one has

$$(2.11) \quad d_m = \underline{b}^T (\underline{I}^* \underline{P})^m \underline{e}_0, \quad m \geq 0,$$

where $\underline{I}^* = \text{diag}\{0, 1, 1, \dots, 1\}$, and $\underline{e}_1^T = (1, 0, 0, \dots, 0)$.

§.3 Computational Procedures

We have seen in Section 2 that all performance measures can be obtained if $\underline{F}_{i,m}^T(+\infty, t)$ and $\underline{f}_{1,m}^T(0+, t)$ are computed. In this section, we develop numerical procedures for evaluating these probabilistic entities. We assume that both $a_n(x)$ and $r_n(x)$ belong to the class of rapidly decreasing functions of Dym and McKean(1972), so that the corresponding Fourier-Laguerre coefficients are also rapidly decreasing, see, Keilson and Nunn(1979). The Laplace transform of $\underline{F}_{i,m}^T(+\infty, t)$ denoted by $\hat{\Phi}_{i,m}^T(s) = \int_0^\infty e^{-st} \underline{F}_{i,m}^T(+\infty, t) dt$ can be found from (1.21) and (1.22) by setting $w = 0+$, that is

$$(3.1) \quad \begin{cases} \hat{\Phi}_{0,m}^T(s) = (1/s) \hat{\varphi}_{0,m}^T(0+, s) (\underline{I} - \underline{\rho}_D(s)), & m \geq 1, \\ \hat{\Phi}_{1,m}^T(s) = (1/s) \hat{\varphi}_{1,m}^T(0+, s) (\underline{I} - \underline{\alpha}_D(s)), & m \geq 0. \end{cases}$$

The entities $\hat{\varphi}_{0,m}^T(0+, s)$ and $\hat{\varphi}_{1,m}^T(0+, s)$ needed here are given in (1.19) and (1.20). The inversion of these transforms require multiple convolutions of matrix functions in the time domain. The matrix Laguerre transform developed by Sumita(1984) provides a computational vehicle for this purpose.

The Laguerre transform, introduced in Keilson and Nunn(1979) and Keilson, Nunn and Sumita(1981) and further studied by Sumita(1981), provides an algorithmic framework for the computer evaluation of multiple convolutions and other continuum operations. The transform based on generalized Fourier series employs the Laguerre functions as a basis, and maps the functions $f(x)$ in L_2 into discrete sequences $(f_n^\#)_{-\infty}^{+\infty}$. Correspondingly, various continuum operations are mapped into lattice operations, thereby providing the desired algorithmic basis. Recently the formalism has been extended to the matrix form for the study of semi-Markov processes by Sumita(1984) which is the crucial numerical tool employed here. For the reader's convenience a concise summary of the matrix Laguerre transform is given in the Appendix. The notation there is employed

throughout the rest of the paper.

Let $a_n(x)$ and $r_n(x)$ have the Laguerre sharp coefficients $(a_{n:k}^\#)_{k=0}^\infty$ and $(r_{n:k}^\#)_{k=0}^\infty$. One then has

$$(3.2) \quad c_n(x) = \int_0^x a_n(x-y)r_n(y)dy \mapsto c_{n:k}^\# = \sum_{j=0}^k a_{n:k-j}^\# r_{n:j}^\#.$$

Let $\underline{z}(x)$ be the matrix function defined by

$$(3.3) \quad \int_0^\infty e^{-st} \underline{z}(x) dx = \underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P}.$$

It can be readily seen that $\underline{z}(x)$ has a sequence of Laguerre sharp coefficient matrices $\underline{z}_k^\# = [z_{k:ij}^\#]$ where

$$(3.4) \quad z_{k:ij}^\# = p_{ij} c_{i:k}^\#, \quad 0 \leq i, j \leq K, \quad k = 0, 1, \dots$$

We note that $a_0(x) = r_0(x) = 0$ so that $a_{0:k}^\# = r_{0:k}^\# = 0$ for $k \geq 0$ and hence $c_{0:k}^\# = 0$, $k \geq 0$. If $\underline{z}^{(m)}(x)$, having the Laguerre sharp coefficient matrices $(\underline{z}_k^\#(m))$, corresponds to $(\underline{\alpha}_D(s) \underline{\rho}_D(s) \underline{P})^m$, one has

$$(3.5) \quad \underline{z}_k^\#(m+1) = \sum_{j=0}^k \underline{z}_{k-j}^\#(m) \underline{z}_j^\#.$$

Hence the Laguerre sharp coefficient vectors of $\underline{f}_{1,m}^T(0+, t)$ can be easily computed using (1.20). Those corresponding to $\underline{f}_{0,m+1}^T(0+, t)$ can then be obtained by convolving the resulting sharp coefficient vectors for $\underline{f}_{1,m}^T(0+, t)$ with the Laguerre sharp coefficient matrices $(\underline{a}_{Dk}^\#)_0^\infty$ for $\underline{a}_D(x)$ corresponding to $\underline{\alpha}_D(s)$. It should be noted that $\underline{a}_{Dk}^\#$ is a diagonal matrices whose n -th diagonal element is $a_{n-1,k}^\#$. Once the Laguerre sharp coefficient vectors are found, the function values of $\underline{f}_{1,m}^T(0, t)$ can be computed straightforwardly following the inversion procedure described in the Appendix.

Evaluation of the function values of $\underline{F}_{i,m}(+\infty, t)$ requires more caution since $F_{i,m,n}(+\infty, t)$ may not be in L_2 . In particular $F_{1,m,0}(+\infty, t)$ converges to a positive constant (see(2.11)) as $t \rightarrow \infty$. From (3.1), one sees that $\underline{F}_{i,m}^T(+\infty, t)$ is differentiable with respect to t for all $m \geq 0$. Let

$$(3.6) \quad \underline{g}_{i,m}^T(t) = \frac{\partial}{\partial t} \underline{F}_{i,m}^T(+\infty, t)$$

and define

$$(3.7) \quad \hat{\underline{g}}_{i,m}^T(s) = \int_0^\infty e^{-st} \underline{g}_{i,m}^T(t) dt.$$

It should be noted that $\hat{\underline{g}}_{i,m}^T(s) = s \hat{\underline{\Phi}}_{i,m}^T(s) - \underline{F}_{i,m}^T(+\infty, 0)$. Clearly $\underline{F}_{i,m}^T(+\infty, 0) = \underline{0}^T$ for $i = 1, m \geq 1$ or $i = 0, m \geq 0$ and $\underline{F}_{1,0}(+\infty, 0) = \underline{b}$. Hence from (3.1) one obtains

$$(3.8) \quad \begin{cases} \hat{\underline{g}}_{0,m}^T(s) = \hat{\underline{\Phi}}_{0,m}^T(0+, s)(\underline{I} - \underline{\rho}_D(s)), & m \geq 1, \\ \hat{\underline{g}}_{1,m}^T(s) = \hat{\underline{\Phi}}_{1,m}^T(0+, s)(\underline{I} - \underline{\alpha}_D(s)) - \delta_{0m} \underline{b}^T, & m \geq 0. \end{cases}$$

Here $\delta_{0m} = 1$ if $m = 0$, $\delta_{0m} = 0$ otherwise. It can be readily seen from (3.8) that $\underline{g}_{i,m}^T(t)$ are Laguerre transformable (see Remark 3.1) and the corresponding Laguerre sharp coefficient vector denoted by $(\underline{g}_{i,m;k}^\#)_{k=0}^\infty$ can be found as before. Let

$$(3.9) \quad \underline{\overline{G}}_{i,m}^T(t) = \int_t^\infty \underline{g}_{i,m}^T(\tau) d\tau.$$

We note from (1.20) that

$$(3.10) \quad \begin{cases} \underline{q}_{0,m}^T = \int_0^\infty \underline{f}_{0,m}^T(0+, t) dt = \underline{b}^T (\underline{I}^* \underline{P})^{m-1} \underline{I}^* \leq \underline{1}^T, & m \geq 1, \\ \underline{q}_{1,m}^T = \int_0^\infty \underline{f}_{1,m}^T(0+, t) dt = \underline{b}^T (\underline{I}^* \underline{P})^m \leq \underline{1}^T, & m \geq 0. \end{cases}$$

Here the n -th component of $\underline{q}_{0,m}^T$ is the probability that upon the occurrence of the m -th failure there are n software errors in the software system. Since there may be positive probability that all software errors are eliminated before reaching the m -th failure, it is

possible to have $\underline{q}_{1,m}^T \mathbf{1} < 1$. The vector $\underline{q}_{1,m}^T$ has a similar probabilistic meaning. One now sees from (3.8) and (3.10) that since $q_{0,m,0} = 0$,

$$(3.11) \quad \begin{cases} \hat{\underline{g}}_{0,m}^T(0) = [0, 0, \dots, 0], & m \geq 1 \\ \hat{\underline{g}}_{1,m}^T(0) = [q_{1,m,0}, 0, \dots, 0] - \delta_{0m} \underline{b}^T, & m \geq 0. \end{cases}$$

Hence $\overline{\underline{G}}_{1,m}^T(0) = \hat{\underline{g}}_{1,m}^T(0) < \infty$ and Equation (3.9) is well defined. By applying one of the operational properties of the matrix Laguerre transform (see (A13)), the Laguerre sharp coefficient vectors of $\overline{\underline{G}}_{1,m}^T(t)$ can be generated from $(\underline{g}_{1,m;k}^\#)_{k=0}^\infty$ for $\underline{g}_{1,m}^T(t)$. One then finally has

$$(3.12) \quad \underline{F}_{1,m}^T(+\infty, t) = \overline{\underline{G}}_{1,m}^T(0) - \overline{\underline{G}}_{1,m}^T(t) + \delta_{m0} \delta_{11} \underline{b}^T.$$

Remark 3.1

We note from (3.8) that $\underline{g}_{1,0}^T(t)$ is a generalized vector function involving the delta function $\delta(x)$. Hence the components of $\underline{g}_{1,0}^T(t)$ are not in L_2 . In a recent paper by Keilson and Sumita(1984), it has been shown that the Laguerre sharp transform exists for any finite signed measure preserving all basic operational properties. The Laguerre sharp coefficients for $\underline{g}_{1,0}^T(t)$ therefore exists.

§.4 Numerical results

In this section, we demonstrate the efficiency of the computational procedures developed in Section 3 through a numerical example. Tables and graphs illustrating numerical results are given at the end of this section. We consider a software system with the following features corresponding to (AS1) through (AS5) of Section 1.

$$(4.1) \quad K = 4.$$

$$(4.3) \quad \underline{b}^t = (0.2, \quad 0.2, \quad 0.2, \quad 0.2, \quad 0.2)$$

$$(4.3) \quad A_0(x) = 0 \text{ and } A_n(x) = \int_0^x \frac{2^{5-n}}{(4-n)!} y^{4-n} e^{-2y} dy, \quad 1 \leq n \leq 4.$$

$$(4.4) \quad R_n(x) = \int_0^x \frac{3^{5-n}}{(4-n)!} y^{4-n} e^{-3y} dy, \quad 1 \leq n \leq 4.$$

$$(4.5) \quad \underline{\underline{P}} = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.700 & 0.150 & 0.100 & 0.030 & 0.020 \\ 0.300 & 0.425 & 0.150 & 0.100 & 0.025 \\ 0.125 & 0.225 & 0.400 & 0.150 & 0.100 \\ 0.075 & 0.125 & 0.200 & 0.400 & 0.200 \end{pmatrix}.$$

We note that the lifetime and the repair time of the system when there are n software errors in the system are the sum of $(5 - n)$ independent exponential random variables with parameter 2 and 3 respectively.

The Laguerre sharp coefficients $(a_n^\#)_0^\infty$ of an exponential function $a_n(x) = \theta e^{-\theta x}$, $\theta > 0$, can be found analytically (see Keilson and Nunn(1981)) as

$$(4.6) \quad a_0^\# = \frac{\theta}{\theta + \frac{1}{2}}; \quad a_n^\# = -\frac{\theta}{(\theta + \frac{1}{2})^2} \left(\frac{\theta - \frac{1}{2}}{\theta + \frac{1}{2}} \right)^{n-1}, \quad n \geq 1.$$

This formula enables one to generate the Laguerre sharp coefficient matrices corresponding to $\underline{\alpha}_D(s)$ and $\underline{\rho}_D(s)$ via discrete convolutions. In actual computation, the first 101

coefficients were used. Using equation (1.17) and (1.18), the Laguerre sharp coefficient vectors of $\underline{f}_{i,m}^T(0+, t)$, $i = 0, 1$, can then be obtained via discrete vector-matrix convolutions. As discussed in Appendix, the moment formula of the matrix Laguerre transform provides a heuristic tool for checking accuracy. Let

$$(4.7) \quad \underline{\mu}_{i,m}^T(k) = \int_0^\infty t^k \underline{f}_{i,m}^T(0+, t) dt, \quad 0 \leq k \leq 2.$$

By differentiating (1.17) and (1.18) with respect to s at $s = 0$, one then finds the following recursion formulas:

$$(4.8) \quad \begin{cases} \underline{\mu}_{1,0}^T(0) = \underline{b}^T \\ \underline{\mu}_{1,m}^T(0) = \underline{\mu}_{0,m}^T(0) \underline{I}^* \underline{P} \\ \underline{\mu}_{0,m}^T(0) = \underline{\mu}_{1,m-1}^T(0) \underline{I}^* \end{cases}$$

$$(4.9) \quad \begin{cases} \underline{\mu}_{1,m}^T(1) = (\underline{\mu}_{0,m}^T(1) \underline{I}^* + \underline{\mu}_{0,m}^T(0) \underline{M}_{RPD}(1)) \underline{P} \\ \underline{\mu}_{1,0}^T(1) = \underline{0} \\ \underline{\mu}_{0,m}^T(1) = \underline{\mu}_{1,m-1}^T(1) \underline{I}^* + \underline{\mu}_{1,m-1}^T(0) \underline{M}_{LFD}(1) \end{cases}$$

$$(4.10) \quad \begin{cases} \underline{\mu}_{1,m}^T(2) = (\underline{\mu}_{0,m}^T(2) \underline{I}^* + 2\underline{\mu}_{0,m}^T(1) \underline{M}_{RPD}(1) + \underline{\mu}_{0,m}^T(0) \underline{M}_{RPD}(2)) \underline{P} \\ \underline{\mu}_{1,0}^T(2) = \underline{0} \\ \underline{\mu}_{0,m}^T(2) = \underline{\mu}_{1,m-1}^T(2) \underline{I}^* + 2\underline{\mu}_{1,m-1}^T(1) \underline{M}_{LFD}(1) + \underline{\mu}_{1,m-1}^T(0) \underline{M}_{LFD}(2) \end{cases}$$

Here $\underline{M}_{LFD}(k) = \text{diag}\{0, \int_0^\infty x^k a_1(x) dx, \dots, \int_0^\infty x^k a_4(x) dx\}$ for $0 \leq k \leq 2$ and $\underline{M}_{RPD}(k)$ is defined similarly for repair time distributions. Using (4.8) through (4.10), $\underline{\mu}_{i,m}^T(k)$ were calculated for $i = 0, 1$, $1 \leq m \leq 41$ and $0 \leq k \leq 2$. These values were then compared with the values obtained from the Laguerre sharp coefficient vectors of $\underline{f}_{i,m}^T(0+, t)$ and the moment formula of (A15). The relative errors were found to be bounded by 1×10^{-10} in this range of i , m , and k , providing excellent accuracy. In Table 4.1, this comparison is exhibited for $i = 1$ and $m = 10$. The value $m = 10$ will be used subsequently for evaluating conditional performance measures.

The Laguerre sharp coefficient vectors of $\underline{g}_{i,m}^T(t)$ in (3.6) can be found from those of $\underline{f}_{i,m}^T(0+,t)$ with one additional discrete vector-matrix convolution using (3.8), which in turn lead to Laguerre dagger coefficient vectors of $\underline{G}_{i,m}^T(t)$ of (3.9) via the operational property of the matrix Laguerre transform given in (A13). The values of $\underline{F}_{i,m}^T(+\infty, t)$ needed can then be calculated from (3.12). Since most of the performance measures involve the expressions $\sum_{m=1}^{\infty} \underline{F}_{0,m}^T(+\infty, t)$ and $\sum_{m=0}^{\infty} \underline{F}_{1,m}^T(+\infty, t)$, we generate the Laguerre coefficient vectors of $\underline{S}_{i,M}^T(t)$ defined by

$$(4.11) \quad \underline{S}_{i,M}^T(t) = \sum_{m=1-M}^M \underline{G}_{i,m}^T(t).$$

Both the Laguerre transform and the tail integral operation are linear and this can be accomplished by merely adding the Laguerre sharp coefficient vectors of $\underline{g}_{i,m}^T(0+,t)$ over m , $0 \leq m \leq M$, and then applying the operational property of (A13) to the resulting sum.

To check the accuracy of the truncated Fourier-Laguerre transform representation of $\underline{S}_{i,M}^T(t)$, one again uses the moment formula of (A15). Let

$$(4.12) \quad \underline{r}_{i,m}^T(k) = \int_0^{\infty} t^k \underline{g}_{i,m}^T(t) dt.$$

By differentiating (3.8) with respect to s at $s = 0$, one finds that:

$$(4.13) \quad \begin{cases} \underline{r}_{0,m}^T(0) = \underline{\mu}_{0,m}^T(0)(\underline{I} - \underline{I}^*) \\ \underline{r}_{1,m}^T(0) = \underline{\mu}_{1,m}^T(0)(\underline{I} - \underline{I}^*) - \delta_{0,m} \underline{b}^T \end{cases}$$

$$(4.14) \quad \begin{cases} \underline{r}_{0,m}^T(1) = \underline{\mu}_{0,m}^T(1)(\underline{I} - \underline{I}^*) - \underline{\mu}_{0,m}^T(0) \underline{M}_{RPD}(1) \\ \underline{r}_{1,m}^T(1) = \underline{\mu}_{1,m}^T(1)(\underline{I} - \underline{I}^*) - \underline{\mu}_{1,m}^T(0) \underline{M}_{LFD}(1) \end{cases}$$

$$(4.15) \quad \begin{cases} \underline{r}_{0,m}^T(2) = \underline{\mu}_{0,m}^T(2)(\underline{I} - \underline{I}^*) - 2\underline{\mu}_{0,m}^T(1) \underline{M}_{RPD}(1) - \underline{\mu}_{0,m}^T(0) \underline{M}_{RPD}(2) \\ \underline{r}_{1,m}^T(2) = \underline{\mu}_{1,m}^T(2)(\underline{I} - \underline{I}^*) - 2\underline{\mu}_{1,m}^T(1) \underline{M}_{LFD}(1) - \underline{\mu}_{1,m}^T(0) \underline{M}_{LFD}(2). \end{cases}$$

It should be noted that

$$(4.16) \quad \int_0^\infty \bar{S}_{i,M}^T(t) dt = \sum_{m=1-i}^M \underline{r}_{i,m}^T(1); \quad 2 \int_0^\infty t \bar{S}_{i,M}^T(t) dt = \sum_{m=1-i}^M \underline{r}_{i,m}^T(2).$$

The zero-th and the first moment of $\bar{S}_{i,M}(t)$ were calculated using the Laguerre sharp coefficient vectors and the moment formula of (A15). These values were then compared with values of $\sum_{m=1-i}^M \underline{r}_{i,m}(1)$ and $\sum_{m=1-i}^M \underline{r}_{i,m}^T(2)$ generated from the recursion formulas in (4.13) through (4.15). Both the relative and absolute errors decrease in M and those errors were found to be bounded by 1×10^{-7} and 3×10^{-8} respectively for $M = 41$. This comparison is summarized in Table 4.2. The truncation level M has been examined using the sequence

$$(4.17) \quad v_M(t) = \sum_{m=0}^M \{E_{0,m+1}^T(+\infty, t) + E_{1,m}^T(+\infty, t)\} \underline{1},$$

which converges to 1 as $M \rightarrow \infty$ for all $t > 0$. For the value $M = 41$, $(1 - v_M(t))$ was found to be bounded by 1×10^{-18} for $0 \leq t \leq 20$, as exhibited in Table 4.3.

The time independent performance measure, d_m , of (2.11) describing the probability of having m software failures before complete debugging is depicted in Figure 4.4. In Figure 4.5, the compound availability/reliability measures $P[I(t) = 1, M(t) = 3, N(t) = n] = F_{1,3,n}(+\infty, t)$ are plotted for $0 \leq n \leq 4$ and $0 \leq t \leq 20$. One observes that $\lim_{t \rightarrow \infty} F_{1,3,0}(+\infty, t) = d_3 = 0.14259$ and $\lim_{t \rightarrow \infty} F_{1,3,n}(+\infty, t) = 0$, $1 \leq n \leq 4$, as expected. The joint probability $P[I(t) = i, N(t) = n]$ of (2.2) are exhibited in Figure 4.6 and 4.7 for $i = 0$, $1 \leq n \leq 4$ and $i = 1$, $0 \leq n \leq 4$, respectively. It may be noted that these joint probabilities are all unimodal for $i = 0$. For $i = 1$, $P[I(t) = 1, N(t) = 0]$ approaches one monotonically as $t \rightarrow \infty$. We note that this joint probability is also the distribution function $S_{N(0),0}(t)$ of the time required for completely eliminating all errors

in the software system, as shown in (2.4). After certain initial period, all other joint probabilities with $i = 1$ decrease monotonically to zero. Figure 4.8 depicts $P[I(t) = i]$ for $i = 0, 1$. This probability with $i = 1$ is unimodal having the peak approximately at time $t = 2$ and decreases to zero as $t \rightarrow \infty$. Correspondingly, $P[I(t) = 0]$ has minimum around $t = 2$ and goes to one as $t \rightarrow \infty$.

We next turn our attention to the software system availability/reliability measures based on the past observation. We assume that at time $t = 20$ a repair for the tenth software failure is completed. In other words we choose $m = 10$ and $t_0 = 20$ in the condition (CD) given in Section 2. The associated probability vector $\underline{\beta}_{t_0, m}$ of (2.5) can be calculated using the Laguerre coefficient vectors of $f_{1,0}^T(0+, t)$ as

(4.18)

$$\underline{\beta}_{t_0, m}^T = (0.4185245993, 0.2485688688, 0.1781236373, 0.1040558505, 0.0507204397, \dots)$$

The counterparts of Figure 4.6 and 4.7 are given in Figure 4.9 and 4.10, describing the joint probability of the system availability, unavailability and the number of errors in the software system at time $t_0 + \tau$ given (CD). It should be noted that the upper curve in Figure 4.10 is the distribution function $G_{N(t_0), 3}(\tau)$ of time until complete debugging under (CD) given in (2.9). The joint probability of $I(t)$ and $M(t)$ given (CD) is plotted in Figure 4.11, corresponding to the formula in (2.8) with $i = 1$ and $j = 3$. Finally the survival function of the time until the next software failure denoted by $\bar{W}_{t_0, m}$ in (2.10) is exhibited in Figure 4.12. We note that $\lim_{\tau \rightarrow \infty} \bar{W}_{t_0, m}(\tau) = \beta_{0, t_0, m} = 0.4185246$, i.e. with this probability all errors in the software system have been eliminated by time $t_0 = 20$ and there will be no software failure.

All computations were done in DEC20 in a time sharing mode using APL as a pro-

programming language. The DEC20 APL implementation is the double precision system which uses a precision of 18 decimal digits. Relevant formulas were usually coded in a straightforward way with no attempt made to optimize the subroutines for speed and accuracy. Evaluation of all performance measures presented in this section required approximately several minutes of CPU time. No evidence of numerical problems were observed through the entire procedure.

Table 4.1 Moment of $f_{i,m,n}(0+,t)$, $i = 1$, $m = 10$

n	0th moment	1st moment	2nd moment
0	0.00356832743960682	0.07572635917463385	1.68068784447817074
	0.00356832743960683	0.07572635917463717	1.68068784447949448
1	0.00210648615538208	0.04302096760275793	0.92238993277688258
	0.00210648615538209	0.04302096760275967	0.92238993277757633
2	0.00153425632614903	0.03061996083817533	0.64343981552739176
	0.00153425632614904	0.03061996083817608	0.64343981552769234
3	0.00091263323266573	0.01777617904156510	0.36542344506696303
	0.00091263323266573	0.01777617904156599	0.36542344506731721
4	0.00044773521740952	0.00868505502745937	0.17791002799323507
	0.00044773521740952	0.00868505502745974	0.17791002799338249

upperline: values via recursive formula

lowerline: values via Laguerre sharp coefficients

Table 4.2 Moment of $\bar{S}_{i,M,n}(t)$, $M = 41$

n	0th moment	1st moment $\times 2$
0	5.2021934394	50.9448926126
	5.2021934383	50.9448921478
1	-1.4812521192	-15.4420556088
	-1.4812521199	-15.4420558898
2	-0.9621601960	-8.1396683807
	-0.9621601971	-8.1396688012
3	-0.5022338213	-3.1119079053
	-0.5022338215	-3.1119079793
4	-0.1756699416	-0.7508504584
	-0.1756699420	-0.7508505963

upperline: values via recursive formula

lowerline: values via Laguerre sharp coefficient

Table 4.3 $1 - v_M(t)$, $M = 41$

time	$1 - v_M(t)$
0	0
4	6.505×10^{-19}
8	6.505×10^{-19}
12	6.505×10^{-19}
16	8.674×10^{-19}
20	6.505×10^{-19}

Figure 4.4
The probability of having m software failures
before complete debugging.

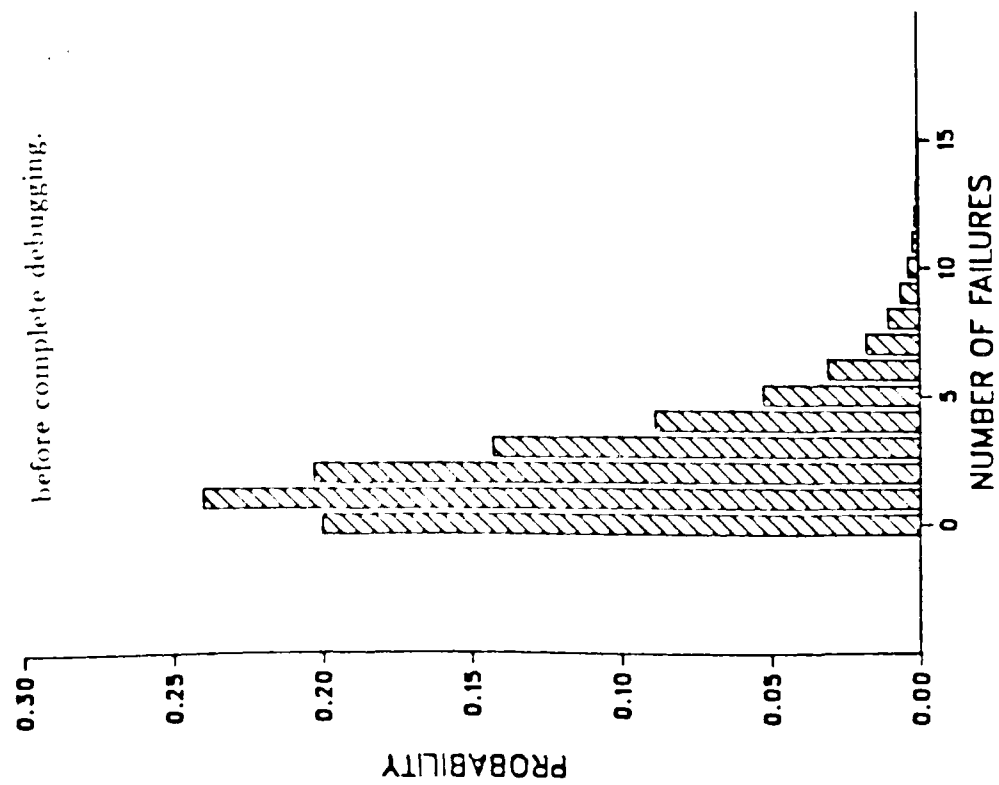


Figure 4.5

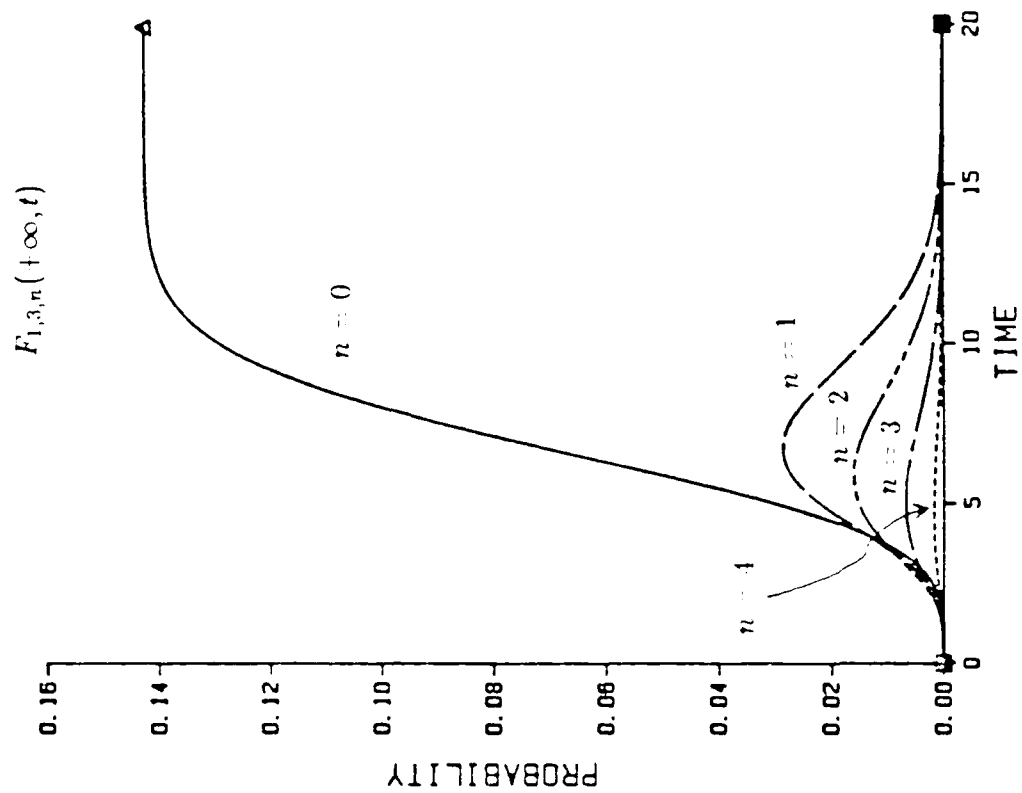


Figure 4.6

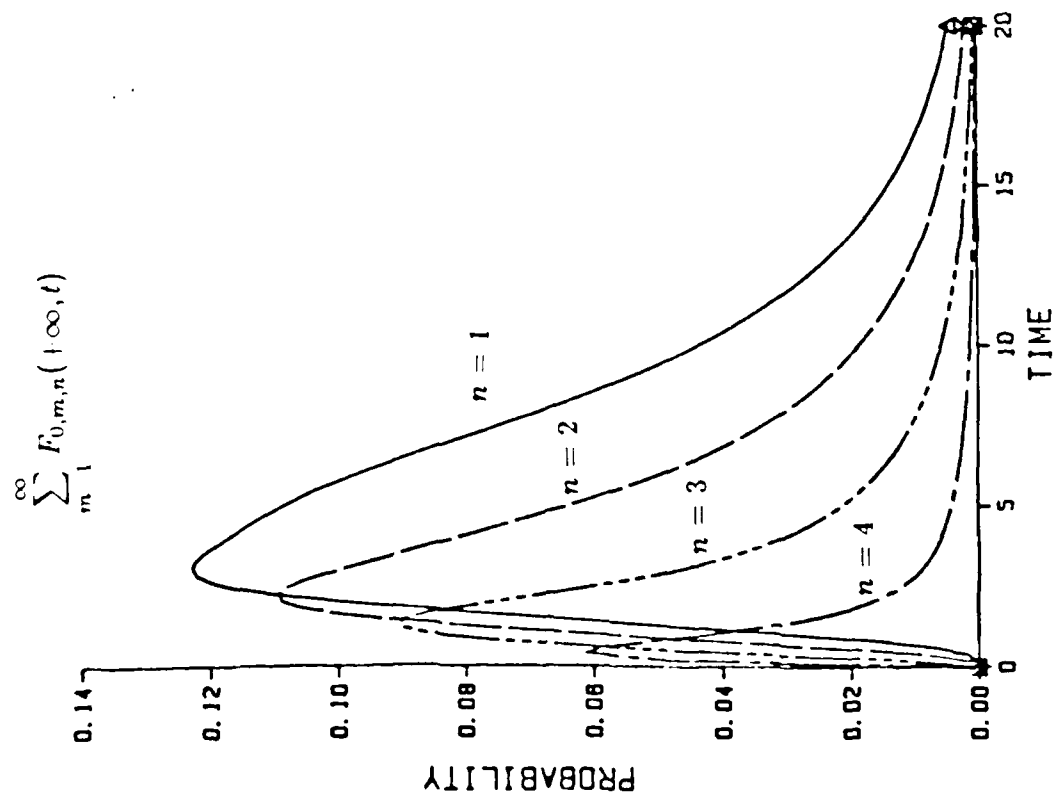


Figure 4.7

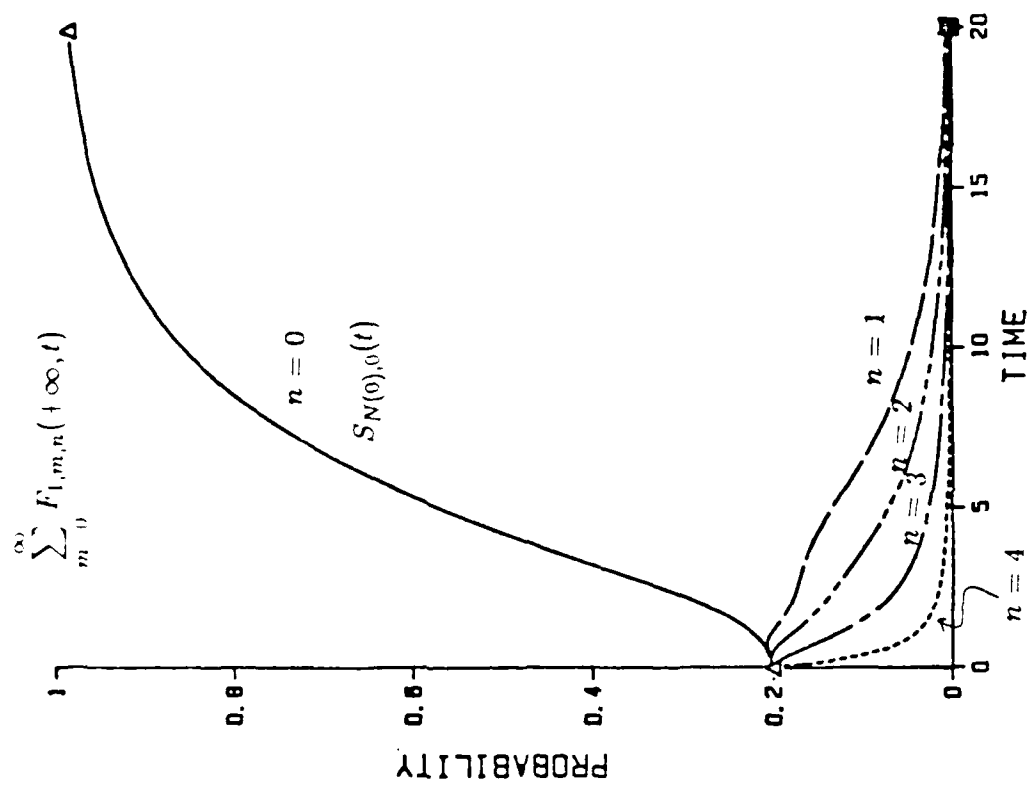


Figure 4.8

$$P[I(t) = i] = \sum_{n=0}^{\infty} F_{i,m,n}(+\infty, t)$$

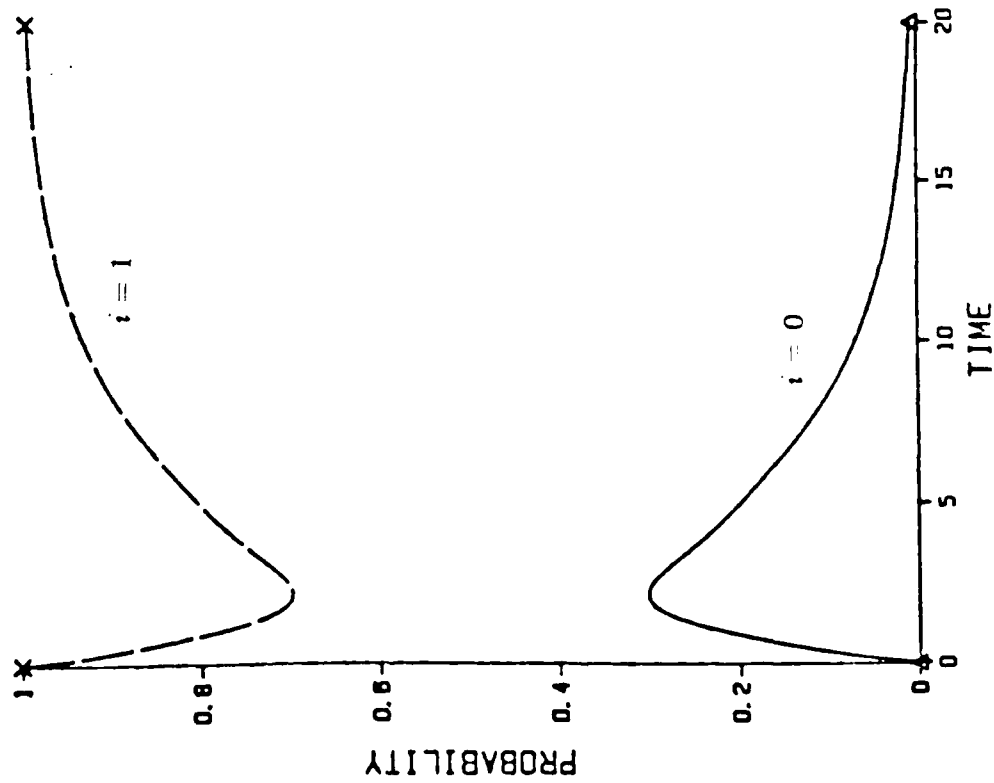


Figure 4.9

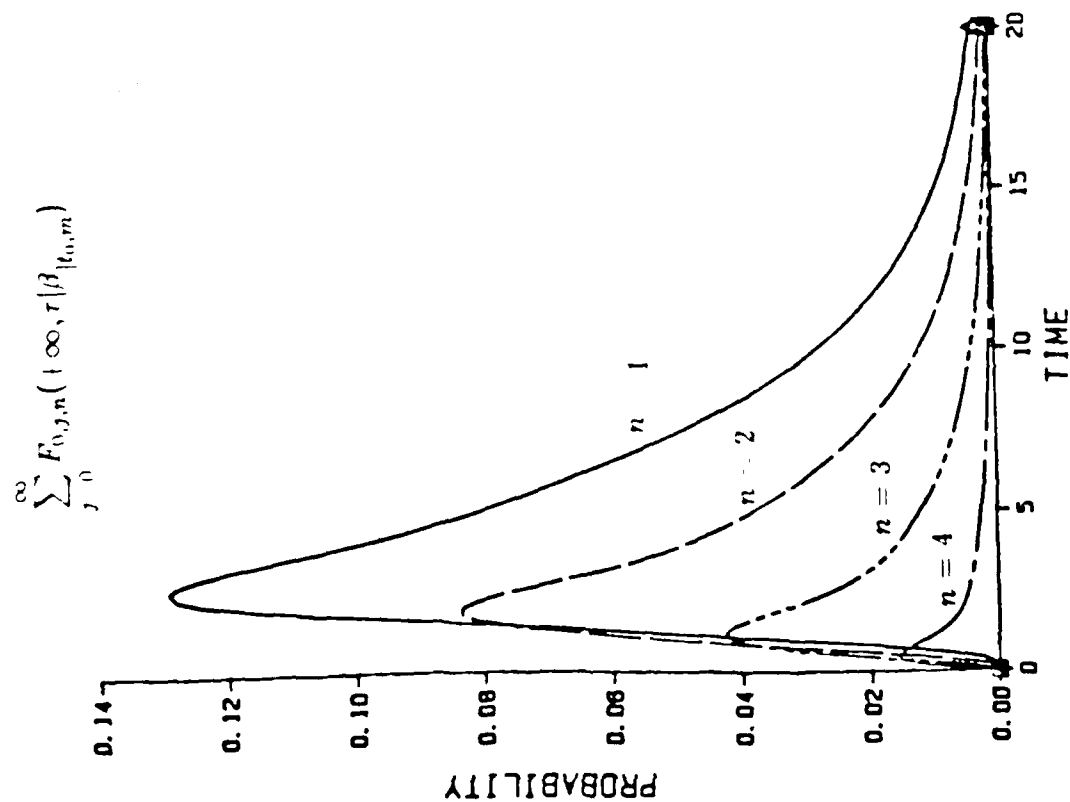


Figure 4.10

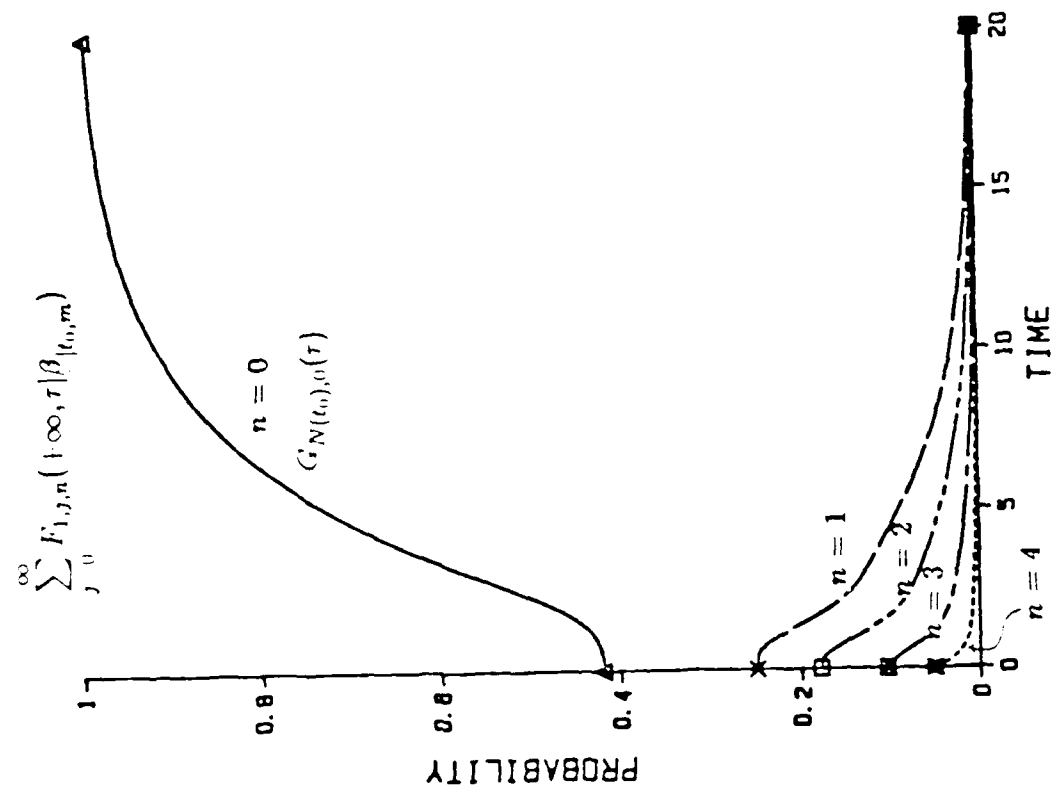


Figure 4.11

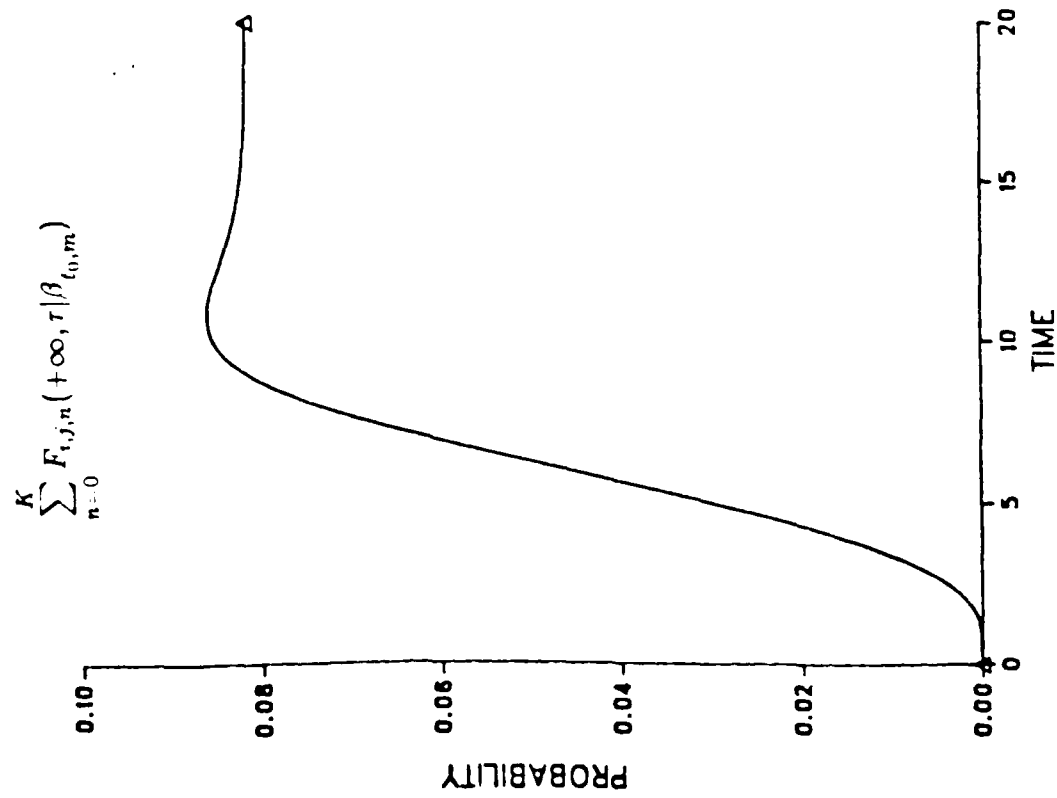
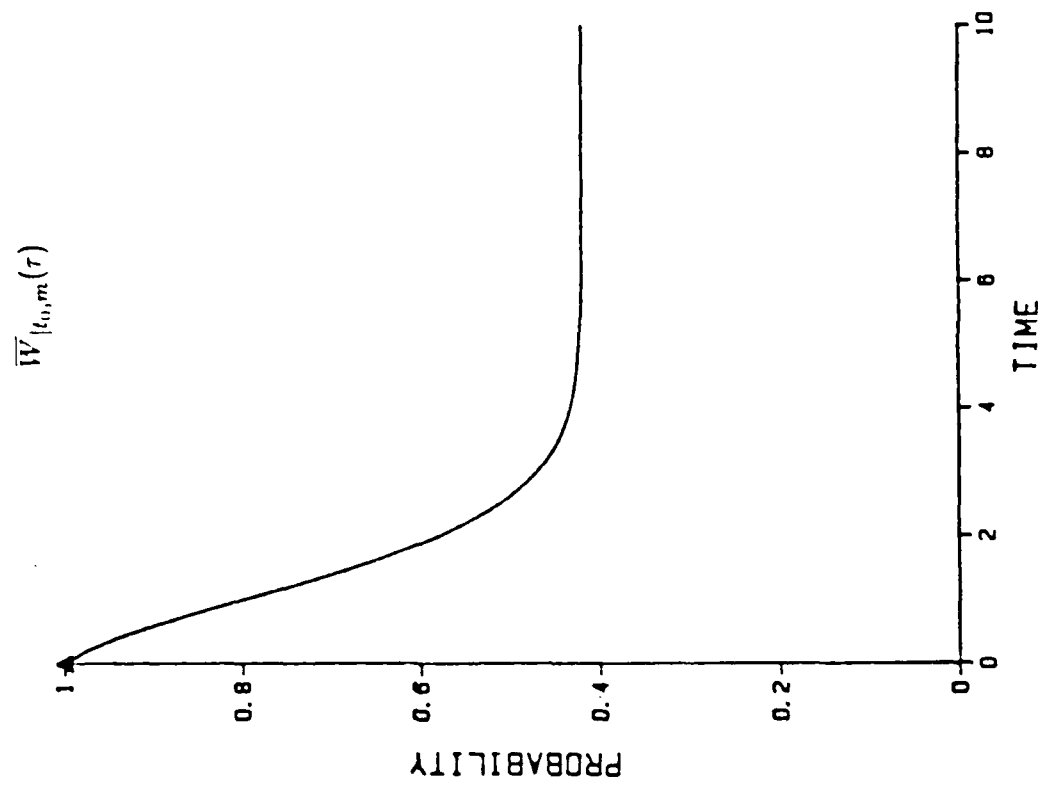


Figure 4.12



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Appendix

In this appendix, we provide a concise summary of the matrix Laguerre transform. The reader is referred to Sumita(1984) for more detailed discussions. The Laguerre polynomial $L_n(x)$ of degree n is defined by the Rodrigues formula

$$(A1) \quad L_n(x) = \left(\frac{1}{n!}\right) e^x \left(\frac{d}{dx}\right)^n x^n e^{-x}, \quad n = 0, 1, 2, \dots$$

The corresponding Laguerre functions $\ell_n(x) = e^{-\frac{1}{2}x} L_n(x)$, $n = 0, 1, 2, \dots$, form an orthogonal basis of $L_2(0, \infty) = \{f : R_+ \rightarrow R : \int_0^\infty f^2(x) dx < \infty\}$. The Laplace transform of $\ell_n(x)$ is given by

$$(A2) \quad \lambda_n(s) = \int_0^\infty e^{-sx} \ell_n(x) dx = \frac{1}{s + \frac{1}{2}} \left(\frac{s - \frac{1}{2}}{s + \frac{1}{2}}\right)^n, \quad n = 0, 1, 2, \dots, \quad \operatorname{Re}(s) > -\frac{1}{2}.$$

We define the linear space \underline{L}_2 of $K \times K$ matrix functions by

$$(A3) \quad \underline{L}_2 = \{\underline{a}(x) = (a_{ij}(x)) ; a_{ij}(x) \in L_2 \text{ for all } 1 \leq i, j \leq K\}.$$

It can be readily seen that the matrix Laguerre functions $\underline{\ell}_n(x) = \ell_n(x) \underline{I}$ provides an orthonormal basis of \underline{L}_2 where \underline{I} is a $K \times K$ identity matrix. Then for any $\underline{a}(x) \in \underline{L}_2$ one has the Fourier-Laguerre series expansion

$$(A4) \quad \underline{a}(x) = \sum_{n=0}^{\infty} \underline{a}_n^+ \underline{\ell}_n(x); \quad \underline{a}_n^+ = \int_0^\infty \underline{a}(x) \underline{\ell}_n(x) dx.$$

The pointwise convergence of (A4) can be assured under certain conditions regarding the smoothness and the rapidly decreasing property of $\underline{a}(x)$. Let

$$(A5) \quad \underline{a}_0^\# = \underline{a}_0^+, \quad \underline{a}_n^\# = \underline{a}_n^+ - \underline{a}_{n-1}^+, \quad n \geq 1,$$

and define the two matrix generating functions $T_{\underline{a}}^+ = \sum_{n=0}^{\infty} \underline{a}_n^+ u^n$ and $T_{\underline{a}}^\# = \sum_{n=0}^{\infty} \underline{a}_n^\# u^n$.

We note that

$$(A6) \quad T_{\underline{a}}^\#(u) = (1 - u) T_{\underline{a}}^+(u), \quad |u| < 1,$$

and

$$(A7) \quad \underline{a}_n^+ = \sum_{m=0}^n \underline{a}_m^{\#}, \quad n = 0, 1, 2, \dots$$

From (A2) and (A4), one sees that

$$(A8) \quad \underline{\alpha}(s) = \int_0^\infty e^{-sx} \underline{a}(x) dx = \sum_{n=0}^\infty \underline{a}_n^+ \frac{1}{s + \frac{1}{2}} \left(\frac{s - \frac{1}{2}}{s + \frac{1}{2}} \right)^n, \quad \text{Re}(s) > -\frac{1}{2}.$$

By letting $u = \frac{1}{s + \frac{1}{2}} \left(\frac{s - \frac{1}{2}}{s + \frac{1}{2}} \right)$, this leads to

$$(A9) \quad T_{\underline{a}}^{\#}(u) = \underline{\alpha} \left(\frac{1}{2} \frac{1+u}{1-u} \right).$$

Equation (A9) is the key formula for the matrix Laguerre transform, providing a bridge between continuum operations and lattice operations. For the matrix convolution $\underline{c}(x) = \int_0^x \underline{a}(x-y) \underline{b}(y) dy$ with $\underline{a}(x), \underline{b}(x) \in \underline{L}_2$, for example, it can be readily seen that

$$(A10) \quad T_{\underline{c}}^{\#}(u) = T_{\underline{a}}^{\#}(u) T_{\underline{b}}^{\#}(u).$$

or equivalently,

$$(A11) \quad \underline{c}_n^{\#} = \sum_{j=0}^n \underline{a}_{n-j}^{\#} \underline{b}_j^{\#}.$$

The matrix Laguerre transform maps matrix functions $\underline{a}(x), \underline{b}(x) \in \underline{L}_2$ into the matrix sequence $(\underline{a}_n^{\#})_0^\infty$ and $(\underline{b}_n^{\#})_0^\infty$. Correspondingly the matrix convolution on continuum is mapped into the lattice convolution. The resulting sharp matrices $(\underline{c}_n^{\#})_0^\infty$ can be converted to the Laguerre dagger coefficient matrices $(\underline{c}_n^{\dagger})_0^\infty$ using (A7). The values $\ell_n(x)$ can be generated efficiently via the recursive formula

$$(A12) \quad \ell_{n+1}(x) = \frac{1}{n+1} [(2n+1-x)\ell_n(x) - n\ell_{n-1}(x)], \quad n \geq 1$$

starting with $\ell_0(x) = e^{-\frac{1}{2}x}$ and $\ell_1(x) = (1-x)e^{-\frac{1}{2}x}$. Hence the Laguerre dagger coefficient matrices $(\underline{c}_n^{\dagger})_0^\infty$ can be inverted back onto continuum via the series representation (A4).

Other continuum operations are mapped into lattice operation in a similar manner. We list only a few operational properties below.

Integration

Let $\underline{g}(x) = \int_x^\infty \underline{a}(y) dy$. Then

$$(A13) \quad \underline{g}_n^\# = -2 \underline{a}_n^\# + 4 \sum_{m=0}^{\infty} (-1)^m \underline{a}_{n+m}^\#.$$

Multiplication by polynomials

Let $\underline{g}(x) = x \underline{a}(x)$. Then

$$(A14) \quad \underline{g}_n^\# = -\Delta^2[(n+1)\underline{a}_{n+1}^\#], \quad n \geq 0,$$

where $\Delta \underline{a}_n = \underline{a}_n - \underline{a}_{n-1}$.

The first two moments of $\underline{a}(x)$ can be also evaluated in terms of $(\underline{a}_n^\#)_{n=0}^\infty$.

Moment formula

$$(A15) \quad \int_0^\infty x^i \underline{a}(x) dx = 4^i \sum_{n=0}^{\infty} (-1)^i n^i \underline{a}_n^\#, \quad 0 \leq i \leq 2.$$

The higher moments can be computed by combining (A14) and (A15) repeatedly. The moment formula (A15) provides a empirical tool for deciding the truncation point of the series representation (A4), when the moment values are known. Extensive numerical experiments suggest that if the truncation point is chosen to satisfy a given accuracy for the moment value in (A15), then the series representation (A4) also satisfies the same accuracy.

It should be noted that for many of matrix functions of interest in applied probability and statistics, the Laguerre coefficient matrices can be obtained either numerically or via certain numerical procedure based on (A9).

END

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